Critical velocities $c/\sqrt{3}$ and $c/\sqrt{2}$ in general theory of relativity

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Abstract

We consider a few thought experiments of radial motion of massive particles in the gravitational fields outside and inside various celestial bodies: Earth, Sun, black hole. All other interactions except gravity are disregarded. For the outside motion there exists a critical value of coordinate velocity $\mathrm{v}_c = c/\sqrt{3}$: particles with $\mathrm{v} < \mathrm{v}_c$ are accelerated by the field, like Newtonian apples, particles with $\mathrm{v} > \mathrm{v}_c$ are decelerated like photons. Particles moving inside a body with constant density have no critical velocity; they are always accelerated. We consider also the motion of a ball inside a tower, when it is thrown from the top (bottom) of the tower and after classically bouncing at the bottom (top) comes back to the original point. The total time of flight is the same in these two cases if the initial proper velocity v_0 is equal to $c/\sqrt{2}$.

1 Introduction. Proper and coordinate velocities

According to the theory of relativity a time interval depends both on the velocity of clocks and on the gravitational potential. There exist infinitely many different coordinates. In the majority of the physical experiments the gravitational potential remains constant in the laboratory and it is convenient to use the so-called proper time τ which is determined by the clocks at rest in the laboratory frame. Maybe this is the reason why sometimes in the literature the proper time is called "genuine", or "physical". A lightminded

person would think that any other time (the coordinate time) is not physical, and thus should not be considered. As a result some people working in the General Theory of Relativity (GTR) consider coordinate-dependent quantities as nonphysical, so to say “second-quality” quantities.

However the coordinate time t is even more important for some problems than the proper time τ. A good example is a particle moving in a static gravitational field, which does not depend on the time coordinate t. The proper time should be used when discussing experiments performed in one and the same gravitational potential (say, on a given floor of a house). However to discuss events happening at different floors observers should introduce corrections which take into account the difference of the gravitational potentials. The coordinate time plays important role in the global positioning system (GPS), where atomic clocks are running on several dozens of the Earth satellites, and in metrology in general [1, 2]. That is why the coordinate quantities are also “genuine”, or “physical”.

The solution of the Einstein equations outside a spherically symmetric massive body was given by Schwarzschild [3] and can be found in all textbooks on GTR (see, for example, [4, 5, 6, 7]). Simultaneously Schwarzschild found the metric inside a spherically symmetric body of constant density [8]. This paper is less known. Below we will compare the behaviour of coordinate velocities in the Schwarzschild metrics [3] and [8]. Some other coordinate systems will be considered at the end of the paper.

As found in papers [9] - [11], in the gravitational field of a spherically symmetric object (the Earth, the Sun, other stars) there exists a critical value of the coordinate speed $\mathrm{v}_c = c/\sqrt{3}$. Particles moving radially with this speed do not accelerate or decelerate in the first order of the Newton constant G – thus, they “ignore” gravity.

For $\mathrm{v} < \mathrm{v}_c$ a falling object accelerates (a famous example is the Newton apple). For $\mathrm{v} > \mathrm{v}_c$ a falling object decelerates (a well known example is the electromagnetic wave, or photon). This deceleration is a reason for the radar echo delay ([4], chapter 8, §7) and the deviation of light by the Sun ([4], chapter 8, §5).

That is why an intermediate critical coordinate velocity v_c should exist.

We will show that outside the Sun $\mathrm{v}_c = c/\sqrt{3}$, while for a particle moving inside the Sun such a critical velocity does not exist: any particle (even photon) accelerates moving to the Sun center.

To study gravitational effects for particles moving inside celestial bodies one should select such particles the non-gravitational interaction of which

with normal matter can be neglected. Neutrinos, neutralinos and mirror particles [12] moving inside the Sun and the Earth satisfy this criterium. Another example – photons, electrons and nucleons moving inside a mirror star. Finally particles moving inside a mine on the Earth also satisfy the above condition.

In section 2 we introduce the necessary notations and obtain one of the main results: $\mathrm{v}_c = 1/\sqrt{3}$ for a particle moving in a weak gravitational field: $r > R \gg r_g$, where r is the particle coordinate, R – the star radius and r_g – the gravitational radius of the star [4] - [7]. Here and in what follows we put $c = 1$, that is why:

$$r_g = 2GM \quad , \tag{1}$$

where M is the mass of the star and G – the gravitational constant.

In section 3 we consider the motion outside a star with strong gravity (neutron star, black hole).

In section 4 we consider the motion inside a star, $r < R$, and demonstrate that both apples and photons are accelerated when freely falling inside a star.

In section 5 the notion of a critical proper (not coordinate!) velocity is illustrated by the following thought experiment: a massive ball is thrown from the top (bottom) of the tower and after elastically bouncing at the bottom (top) comes back after time τ_+ (τ_-). The initial proper velocities in both cases are equal v_0. For $v_0 \ll 1$ we have $\tau_+ < \tau_-$, while for $v_0 = 1$ we have $\tau_+ > \tau_-$. We found that $\tau_+ = \tau_-$ for $v_0 = v_c \equiv 1/\sqrt{2}$.

In section 6 other coordinate systems are considered (harmonic and isotropic) which differ from the Schwarzschild one for $r \sim r_g$, but describe flat space for $r \gg r_g$. In weak gravitational fields ($r/r_g \gg 1$) the critical coordinate velocity in these coordinates is also $1/\sqrt{3}$ (plus corrections $\sim (r_g/r)^2 \ll 1$). Concerning $v_c = 1/\sqrt{2}$ it is invariant under a change of coordinates.

2 Derivation of $\mathrm{v}_c = 1/\sqrt{3}$ for $r > R$

For a radial motion ($\dot\theta = \dot\varphi = 0$) the interval looks like [3]:

$$ds^2 = g_{00}dt^2 - g_{rr}dr^2 \equiv d\tau^2 - dl^2 \quad , \tag{2}$$

where

$$g_{00} = 1 - \frac{r_g}{r} \quad , \tag{3}$$

$$g_{rr} = \left(1 - \frac{r_g}{r}\right)^{-1} = \frac{1}{g_{00}} \quad . \tag{4}$$

The so-called local (or proper) velocity v and coordinate velocity v are the main subject of this paper. The local velocity is measured by a local observer with the help of a ruler and a clock which he has in hands. Doing this measurement the local observer ignores that time is contracted and radius is stretched by gravity:

$$v = \frac{dl}{d\tau} \quad , \tag{5}$$

where the local (proper) coordinates l and τ are defined in Eq. (2).

The same observer will measure the coordinate velocity if he takes into account that his rulers and clocks are influenced by gravity and uses the coordinates r and t:

$$\mathrm{v} = \frac{dr}{dt} \quad , \tag{6}$$

$$\mathrm{v} = v\left(\frac{g_{00}}{g_{rr}}\right)^{1/2} \quad , \tag{7}$$

$$\mathrm{v} = v g_{00} \quad . \tag{8}$$

Here the last equation follows from (7) and (4). From (3) and (4) it follows that at $r = \infty \quad g_{00}(\infty) = g_{rr}(\infty) = 1$, that is why the coordinate velocity coincides with the velocity measured by an observer who resides infinitely far from the source of gravity (star).

When a body moves radially from $r = a$ to $r = b$ with a coordinate velocity $\mathrm{v}(r)$, the following coordinate time elapses:

$$t = \int_a^b \frac{dr}{\mathrm{v}} \quad . \tag{9}$$

That is why to calculate the echo delay for a radar located far from the star the coordinate velocity v is relevant.

For a particle freely moving in a static gravitational field the conserved energy can be introduced:

$$E = \frac{m\sqrt{g_{00}}}{\sqrt{1 - v^2}} \quad , \tag{10}$$

(see [5], eq.(88.9)). The law of energy conservation,

$$E(r = \infty) = E(r) \quad , \tag{11}$$

allows to determine $v(r)$:

$$1 - v^2 = (1 - \mathrm{v}_\infty^2) g_{00} \quad . \tag{12}$$

Thus for a falling massive particle the proper velocity v increases, while it does not change for a photon: $v_\gamma = 1$. $\mathrm{v}(r)$ has a more complicated behavior:

$$1 - \frac{\mathrm{v}^2}{g_{00}^2} = (1 - \mathrm{v}_\infty^2) g_{00} \quad . \tag{13}$$

For $r > R \gg r_g$, when the gravitational field is weak, from (13) and (4) we get:

$$\mathrm{v}^2 = \mathrm{v}_\infty^2 + \frac{r_g}{r}(1 - 3\mathrm{v}_\infty^2) \quad . \tag{14}$$

For $\mathrm{v}_\infty \ll 1$ a well-known law for the motion of nonrelativistic particle in gravitational field follows:

$$\mathrm{v}^2 = \mathrm{v}_\infty^2 + \frac{2MG}{r} \quad . \tag{15}$$

For $\mathrm{v}_\infty = \mathrm{v}_c = 1/\sqrt{3}$ the coordinate velocity does not depend on r: $\mathrm{v} = \mathrm{v}_\infty$.

For $\mathrm{v}_\infty > \mathrm{v}_c$ a freely falling particle slows down!

Concerning a nonradial motion one easily finds that a massive body is deflected by the Sun at any speed, and the deflection angle θ is larger than that of light θ_γ:

$$\theta = \frac{1}{2}\theta_\gamma (1 + \mathrm{v}_\infty^{-2}) \quad , \tag{16}$$

where

$$\theta_\gamma = \frac{2r_g}{R} \quad . \tag{17}$$

Here R is the minimum distance between photon, radiated by the distant star, and the Sun (see [7], problem 15.9, Eq. 13).

3 Variation of v outside a black hole

For a strong gravitational field $r_g/r \sim 1$ we get:

$$1 - \frac{\mathrm{v}^2}{(1 - \frac{r_g}{r})^2} = (1 - \mathrm{v}_\infty^2)(1 - \frac{r_g}{r}) \quad , \tag{18}$$

$$\mathrm{v}^2 = [1-(1-\mathrm{v}_\infty^2)(1-\frac{r_g}{r})](1-\frac{r_g}{r})^2 \ . \tag{19}$$

For $\mathrm{v}_\infty = 1/\sqrt{3}$

$$\mathrm{v}^2 = \frac{1}{3}(1+\frac{2r_g}{r})(1-\frac{r_g}{r})^2 \ . \tag{20}$$

For large values of r ($r_g/r \ll 1$) one can neglect $(r_g/r)^2$ and $(r_g/r)^3$ terms, and from (18), (20) it follows, that $\mathrm{v}^2 = 1/3$ and does not depend on r. For $r_g/r \sim 1$ the behaviour of $\mathrm{v}(r)$ changes; in particular it follows from (19) that $\mathrm{v}(r_g) = 0$ for any initial value v_∞.

For $\mathrm{v}_\infty > 1/\sqrt{3}$ the coordinate velocity monotonically decreases when r decreases. Thus even moderately relativistic particles behave like a photon.

For $\mathrm{v}_\infty < 1/\sqrt{3}$ a particle starts to accelerate and reaches a maximum coordinate velocity:

$$\mathrm{v}_{max} = \frac{2}{3\sqrt{3}} \cdot \frac{1}{(1-\mathrm{v}_\infty^2)} \tag{21}$$

at

$$r_{max} \equiv r(\mathrm{v}_{max}) = \frac{3(1-\mathrm{v}_\infty^2)}{(1-3\mathrm{v}_\infty^2)} r_g \tag{22}$$

After this it starts to slow down, like a photon. Let us note that

$$v(\mathrm{v}_{max}) = 1/\sqrt{3} \ . \tag{23}$$

Thus the proper velocity equals $1/\sqrt{3}$ at zero coordinate acceleration.

4 Variation of v inside a star

As already stated in the Introduction we ignore all interactions except gravitation.

The metric inside a celestial body of constant density (oversimplified model of a star) was found by Schwarzchield [8] (see [4], section 11, §6).

For a radial motion:

$$g_{00} = \frac{1}{4}\left[3\sqrt{1-\frac{r_g}{R}} - \sqrt{1-\frac{r_g r^2}{R^3}}\right]^2 \ , \tag{24}$$

$$g_{rr} = \left(1-\frac{r_g r^2}{R^3}\right)^{-1} \ , \tag{25}$$

where R is the radius of the star. Now $g_{rr} \neq 1/g_{00}$ and while Eq.(7) is still valid, Eq.(8) is not satisfied. For a weak field, when $r_g/R \ll 1$ (let us note that the condition $r_g/R < 1$ is necessary for star stability),

$$g_{00} = 1 - \frac{3}{2}\frac{r_g}{R} + \frac{1}{2}\frac{r_g r^2}{R^3} \quad , \tag{26}$$

$$\begin{aligned} \mathrm{v} &= v\left(\frac{g_{00}}{g_{rr}}\right)^{1/2} = v\left[1 - \frac{3}{2}\frac{r_g}{R} - \frac{1}{2}\frac{r_g r^2}{R^3}\right]^{1/2} = \\ &= v\left(1 - \frac{3}{4}\frac{r_g}{R} - \frac{1}{4}\frac{r_g r^2}{R^3}\right) \quad . \end{aligned} \tag{27}$$

We observe that even photons start to accelerate when they cross the surface of a mirror star. (At the boundary $r = R$ the coordinate acceleration of photons changes sign).

5 A thought experiment in a tower and the critical proper velocity

It is clear that the existence of the critical velocity v_c is not connected with the infinite distance between the observer and the gravitating body. Let us imagine a tower at the Earth surface and an experimentalist with a clock having a metal ball and a metal mirror which reflects this ball elastically. A series of experiments is performed. Each experiment consists of two parts. In the first part the experimentalist throws the ball from the top of the tower with initial proper speed $v_0 = dl^+/d\tau^+$. The ball is reflected by the steel mirror situated at the bottom of the tower and bounces back to the top. The local time of this flight τ^+ is measured at the top of the tower. In the second part the experimentalist throws a ball from the bottom of the tower upwards with initial velocity $v_0 = dl^-/d\tau^-$. The ball is reflected by the mirror situated at the top of the tower and the time interval of the flight τ^- is measured. For nonrelativistic velocities $v_0 \ll c$ we evidently get $\tau^+ < \tau^-$. For photons $v_0 = 1$ and $\tau^+ > \tau^-$. The times τ^+ and τ^- are equal for $v_0 = v_c = 1/\sqrt{2}$. Thus, for $1/\sqrt{2} < v_0 \leq 1$ the ball is quicker when thrown from the bottom. This puzzling result originates from the time delay of the clock in a gravitational field; the same reason, according to which the

photons look redshifted when they move from the bottom of the tower to the top (the famous Pound-Rebka experiment).

Deriving the value of the critical velocity $v_c = 1/\sqrt{2}$ we will see that its value is universal: it does not depend on the value of the gravitational potential and is valid for strong fields as well. The only condition is: $h/r \ll 1 - r_g/r$, where h is the height of the tower. Thus the tower should not be too high (and tidal forces should not destroy the tower, the ball and the experimentalist).

Let us get formulas which illustrate our statements.

The local time interval for the observer at the top of the tower is:

$$d\tau^{+^2} = g_{00}^{+} dt^2 \equiv (1 - \frac{r_g}{r^+}) dt^2 \ , \tag{28}$$

where r^+ is the radial coordinate of the top of the tower in the Schwarzschield coordinates. The time of flight of the ball according to the clock of this observer is:

$$\tau^+ = 2 \int \frac{dr}{dr/d\tau^+} = \frac{2}{\sqrt{1 - \frac{r_g}{r^+}}} \int \frac{dr}{\frac{1}{(1-\frac{r_g}{r^+})} \frac{dr}{dt}} \ . \tag{29}$$

This result immediately follows from Eqs.(9) and (28).

To find the expression for dr/dt let us write the expression for the local speed (which is coordinate invariant):

$$v = \left(\frac{g_{rr}}{g_{00}} \right)^{1/2} \frac{dr}{dt} = \frac{1}{1 - \frac{r_g}{r}} \frac{dr}{dt} \ , \tag{30}$$

and apply the energy conservation equation:

$$\frac{1 - v_0^2}{1 - \frac{r_g}{r^+}} = \frac{1 - v^2}{1 - \frac{r_g}{r}} \ , \tag{31}$$

where v_0 is the initial speed of the ball. From Eqs.(30) and (31) for the square of the denominator of the integral in (29) we get:

$$\frac{1}{(1 - \frac{r_g}{r^+})^2} \left(\frac{dr}{dt} \right)^2 = \left(\frac{1 - \frac{r_g}{r}}{1 - \frac{r_g}{r^+}} \right)^2 \left[1 - \frac{1 - \frac{r_g}{r}}{1 - \frac{r_g}{r^+}} (1 - v_0^2) \right] \ . \tag{32}$$

Expressing $r = r^+ - \varepsilon$ (where ε varies from zero to h, h is the height of the tower in Schwarzschield coordinates) and expanding (32) with respect to

ε we obtain:

$$\frac{1}{(1-\frac{r_g}{r^+})^2}\left(\frac{dr}{dt}\right)^2 = v_0^2 + \frac{r_g}{{r^+}^2(1-\frac{r_g}{r^+})}\varepsilon(1-3v_0^2) \ , \qquad (33)$$

where the weakness of gravitational field is unnecessary and the only requirement is: $h \ll r^+ - r_g$.

Substituting Eq.(33) into Eq.(29) we obtain:

$$\tau^+ = \frac{2}{\sqrt{1-\frac{r_g}{r^+}}}\int\limits_0^h \frac{d\varepsilon}{v_0\left[1+\frac{r_g(1-3v_0^2)}{2{r^+}^2(1-\frac{r_g}{r^+})v_0^2}\varepsilon\right]} \ , \qquad (34)$$

where we suppose that $v_0^2 \gg hr_g/{r^+}^2 = 2gh$.

The same derivation applied for observer at the bottom of the tower leads to the following expression for the time of flight of the ball thrown by him:

$$\tau^- = \frac{2}{\sqrt{1-\frac{r_g}{r^-}}}\int\limits_0^h \frac{d\varepsilon}{v_0\left[1-\frac{r_g(1-3v_0^2)}{2{r^-}^2(1-\frac{r_g}{r^-})v_0^2}\varepsilon\right]} \ , \qquad (35)$$

where $r^- = r^+ - h$. For $v_0 = \mathrm{v}_c = 1/\sqrt{3}$ integrals in Eqs.(34) and (35) coincide, though τ^+ will be still smaller than τ^- because of the difference of the factors which multiply integrals. To find the critical velocity let us expand the expressions in eqs.(34) and (35) with respect to ε and calculate the integrals:

$$\tau^+ = \frac{2h}{v_0\sqrt{1-\frac{r_g}{r^+}}} - \frac{h^2}{v_0\sqrt{1-\frac{r_g}{r^+}}}\frac{r_g(1-3v_0^2)}{2{r^+}^2(1-\frac{r_g}{r^+})v_0^2} \ , \qquad (36)$$

$$\tau^- = \frac{2h}{v_0\sqrt{1-\frac{r_g}{r^-}}} + \frac{h^2}{v_0\sqrt{1-\frac{r_g}{r^-}}}\frac{r_g(1-3v_0^2)}{2{r^-}^2(1-\frac{r_g}{r^-})v_0^2} \ .$$

Now we easily observe that these time intervals coincide for $v_0 \equiv v_c = 1/\sqrt{2}$:

$$\tau^+ = \tau^- = \frac{2h}{v_c\sqrt{1-\frac{2r_g}{r^+ + r^-}}} \ . \qquad (37)$$

Performing an analogous experiment in a mine, an experimentalist will find that v_c is absent – see Section 4.

6 Isotropic and harmonic coordinates

In this section we will demonstrate that the critical velocity v_c equals $1/\sqrt{3}$ in isotropic, harmonic and all other asymptotically flat coordinates, related with Schwarzschild coordinates by the following relations: $t' = t$, $r' = r + \text{const} + O(r_g^2/r^2)$.

According to well-known formulas for the interval in isotropic coordinates we have ([4], chapter 8, §§1-3)

$$ds^2 = \frac{(1 - MG/2\rho)^2}{(1 + MG/2\rho)^2} dt^2 - \left(1 + \frac{MG}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2) \ .$$

Thus in a weak field at $\rho \to \infty$ and for a radial motion $(d\theta = 0,\ d\phi = 0)$ we have:

$$ds^2 = (1 - 2GM/\rho)dt^2 - (1 + 2GM/\rho)d\rho^2 \ .$$

In harmonic coordinates:

$$ds^2 = \left(\frac{1 - MG/R}{1 + MG/R}\right) dt^2 - \left(1 + \frac{MG}{R}\right)^2 d\mathbf{X}^2 - \left(\frac{1 + MG/R}{1 - MG/R}\right) \frac{M^2G^2}{R^4} (\mathbf{X} \cdot d\mathbf{X})^2 \ .$$

where

$$X_1 = R \sin\theta \cos\phi, \quad X_2 = R \sin\theta \sin\phi, \quad X_3 = R \cos\theta \ ,$$

and $R^2 \equiv \mathbf{X}^2$. Thus at $d\theta = 0,\ d\phi = 0$:

$$ds^2 = \left(\frac{1 - MG/R}{1 + MG/R}\right) dt^2 - \left[\left(1 + \frac{MG}{R}\right)^2 + \left(\frac{1 + MG/R}{1 - MG/R}\right) \frac{M^2G^2}{R^2}\right] dR^2 \ .$$

In the limit $R \to \infty$

$$ds^2 = (1 - 2GM/R)dt^2 - (1 + 2GM/R)dR^2 \ .$$

with the accuracy $O(r_g^2/R^2)$. Thus with this accuracy the metric is the same in the Schwarzschield, isotropic and harmonic coordinates.

Performing a derivation analogous to that of eqs. (18), (19) for arbitrary coordinates (r, t) in a spherically symmetric static metric, where for a radial motion we can write

$$ds^2 = g_{00}(r)dt^2 - g_{rr}(r)dr^2 \ ,$$

and generalizing this derivation to the case when the initial proper velocity v_0 is determined not at infinity, but at an arbitrary $r = r_0$, we get:

$$1 - \mathrm{v}^2 \frac{g_{rr}(r)}{g_{00}(r)} = (1 - v_0^2) \frac{g_{00}(r)}{g_{00}(r_0)} \ , \tag{38}$$

and

$$\mathrm{v}^2 = \left[1 - (1 - v_0^2) \frac{g_{00}(r)}{g_{00}(r_0)} \right] \frac{g_{00}(r)}{g_{rr}(r)} \ . \tag{39}$$

In weak fields:

$$g_{rr}(r) = \frac{1}{g_{00}(r)} + O\left(\frac{r_g^2}{r^2} \right) \ ,$$

the coordinate velocity is:

$$\mathrm{v}^2 = \left[1 - (1 - v_0^2) \frac{g_{00}(r)}{g_{00}(r_0)} \right] g_{00}^2(r) \ . \tag{40}$$

One easily observes that the coordinate acceleration is zero when the local velocity $v_0 = 1/\sqrt{3}$ in all coordinates considered by us. For these coordinates at infinity $\mathrm{v} = v$, and $\mathrm{v}_c = 1/\sqrt{3}$ at $r_0 = \infty$, precisely as stated in the beginning of this section.

7 Conclusion and acknowledgements

We demonstrated that in a spherically symmetric gravitational field two critical velocities exist: proper $v_c = 1/\sqrt{2}$ and coordinate $\mathrm{v}_c = 1/\sqrt{3}$. The first is by definition invariant under change of coordinates; the second is with high accuracy the same in the Schwarzschield, isotropic and harmonic coordinates, which are widely used to synchronize clocks on satellites, to analyze the double neutron stars motion and for other relativistic astrophysical objects. We criticized the terminology according to which the proper time is called "physical" or "genuine".

The existence of critical velocities $c/\sqrt{3}$ and $c/\sqrt{2}$ does not contradict the unique role played by light velocity c in the relativistic domain. However the acquaintance with the critical velocities in Schwarzschield coordinates will help the reader to study better the general theory of relativity since these coordinates are "sensible, useful and frequently used" (see [6], v.2, p.526).

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